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DUAL ESTIMATES IN PROBLEMS OF DESIGN OF ELASTIC STRUCTURES†

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The problem of minimizing the volume of two- and three-dimensional structures, subject to certain stress constraints, which are known as the conditions of strength theory and are used in practice for various materials, is considered. The control is achieved by adjusting the shape of the boundary. Cavities inside the design region are allowed, and the shape of the cavities is also optimized. Dual problems, constructed for such optimal design problems, can be used for estimates of optimal or nearly optimal designs. Examples of dual estimates for three optimal design problems are considered.

1. STATEMENT OF THE PROBLEM

WE HAVE previously introduced [1] the notions of the design region and the feasible region, and proved existence theorems for the first and second variations of the displacements of the elastic region. We denote by $O(s, \lambda)$ the set of feasible regions $\Omega \subset \Omega^\circ$, where Ω° is the design region (here $0<\lambda<1$ and s is an integer characterizing the smoothness of the boundary Γ of the region Ω [1]).

Let us formulate the optimal design problem. Suppose we are given the shear modulus μ, Poisson's ratio ν, and the yield point σ_0 of the material, the external load vector $\mathbf{F}$ acting on the part of the boundary $\Gamma_F{}^\circ$, and the section of the boundary $\Gamma_u{}^\circ$, where the displacements of the elastic region are zero. It is required to find

$$\inf J(\mathbf{u}), \quad J = \int_\Omega dx, \quad \forall \Omega \in O(s, \lambda) \tag{1.1}$$

where $\mathbf{u} = u_i \mathbf{e}_i$ is the solution of the integral identity

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$$\int_{\Omega} A(\mathbf{u},\mathbf{v})\,dx - \int_{\Gamma_F} F_i v_i d\Gamma = 0, \quad \forall \mathbf{v} \in V(\Omega) \tag{1.2}$$

$$V(\Omega) = \{\mathbf{v} = v_i(\mathbf{x})\mathbf{e}_i \mid v_i \in W_2^{(1)}(\Omega), \quad v_i(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma_u\}$$
$$\mathbf{x} = x_i \mathbf{e}_i, \quad A(\mathbf{u},\mathbf{v}) = a_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{v})$$
$$\varepsilon_{kl}(\mathbf{v}) = (\partial v_k/\partial x_l + \partial v_l/\partial x_k)/2$$

Here ε_{kl} are the elements of the strain tensor, $\mathbf{e}_i$ are the unit vectors of the Cartesian system of coordinates, $A(\mathbf{u},\mathbf{u})$ is twice the specific energy of elastic strain, a_{ijkl} are the components of the elastic constant tensor of the material and $W_2^{(1)}(\Omega)$ is the space of Sobolev's functions [2]. Here and below we adopt the convention of summation from 1 to N ($N = 2$ or 3) over the repeating indices i, j, k, l in products. In the region Ω, the displacements $\mathbf{u}$ defined by the integral identity (1.2) should satisfy the constraints

$$f(\boldsymbol{\sigma}) \leqslant f_0 \quad (\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}) = \sigma_{ij}(\mathbf{u})\mathbf{e}_i\mathbf{e}_j \quad \sigma_{ij} = a_{ijkl}\varepsilon_{kl}) \tag{1.3}$$

where $\boldsymbol{\sigma}$ is the stress tensor. The function f is either quadratic in the components of the tensor $\boldsymbol{\sigma}$ or piecewise-linear in σ_i—the principal stresses of the tensor $\boldsymbol{\sigma}(\mathbf{u})$ [3], and f_0 is a constant that depends on the elastic constants of the material and on σ_0.

The constraints (1.3) suggest defining the set of functions

$$V_0(\Omega) = \{\mathbf{u} \in V(\Omega) \mid f(\boldsymbol{\sigma}(\mathbf{u})) \leqslant f_0\} \tag{1.4}$$

2. THE DUAL ESTIMATE

To construct the dual problem, we form the Lagrange functional [4]

$$M(\mathbf{u},\mathbf{v}) = \int_{\Omega} [1 + A(\mathbf{u},\mathbf{v})]\,dx - \int_{\Gamma_F} F_i v_i d\Gamma \tag{2.1}$$
$$\forall \Omega \in O(s,\lambda), \quad \forall \mathbf{u} \in V_0(\Omega), \quad \forall \mathbf{v} \in V(\Omega)$$

The functional

$$M_0(\mathbf{u}) = \sup M(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{v} \in V(\Omega)$$

obviously takes finite values, equal to $\operatorname{mes}\Omega$, only for $\mathbf{u}$ that satisfy the integral identity (1.2).

Now define the functional

$$M_0(\mathbf{v}) = \inf M(\mathbf{u},\mathbf{v}), \quad \forall \varepsilon_{ij}(\mathbf{u}) \in L_2(\Omega), \quad \forall \Omega \in \Omega^\circ \tag{2.2}$$

Here we do not assume that $\varepsilon_{ij}(\mathbf{u})$ is generated by some displacement field $\mathbf{u}$. Any function from $L_2(\Omega)$ may be taken as $\varepsilon_{ij}(\mathbf{u})$. The set Ω is only assumed to be measurable and is not necessarily a region. We have the following chain of inequalities:

$$M_0(\mathbf{v}) \leqslant \sup M_0(\mathbf{v}) = \sup\inf M(\mathbf{u},\mathbf{v}) \leqslant$$
$$\leqslant \sup{}^\circ \inf{}^\circ M(\mathbf{u},\mathbf{v}) \leqslant \inf{}^\circ \sup M(\mathbf{u},\mathbf{v}) = \inf{}^\circ M^\circ(\mathbf{u}) \tag{2.3}$$

Here sup is over all $\mathbf{v} \in V(\Omega^\circ)$, inf is over all $\varepsilon_{ij}(\mathbf{u}) \in L_2(\Omega)$, $\Omega \subset \Omega^\circ$, sup° stands for sup over all $\mathbf{v} \in V_0(\Omega)$, and inf° stands for inf over all $\mathbf{u} \in V_0(\Omega)$, $\Omega \in O(s,\lambda)$.

The first inequality in (2.3) is obvious; the third inequality follows from the interchangeability of sup and inf. Let us elucidate the second inequality. Any function $\mathbf{v} \in V(\Omega)$ for $\Omega \in O(s,\lambda)$ can be continued in the region Ω° and conversely $\forall \mathbf{v} \in V(\Omega^0)$ can be restricted to a measurable set $\Omega \subset \Omega^\circ$. In the second inequality, inf° is over a narrower class of functions $\mathbf{u}$ than inf on the left-hand side, which proves the inequality.

From inequality (2.3) it follows that $\forall \mathbf{v} \in V(\Omega^\circ)$, such that $M_0(\mathbf{v}) > 0$, and also the solution of the problem

$$\sup M_0(\mathbf{v}), \quad \forall \mathbf{v} \in V(\Omega^\circ)$$

are the dual estimates of the problem

$$\inf M^\circ(\mathbf{u}),\ \ \forall \mathbf{u}\in V_0(\Omega),\ \ \forall\Omega\in O(s,\lambda)$$

which is equivalent to problem (1.1).

To construct the functional $M_0(\mathbf{v})$ we need to consider the problem of minimizing the integrand of the first integral in (2.1),

$$\inf\chi(\boldsymbol\sigma),\ \ f(\boldsymbol\sigma)\leqslant f_0,\ \ \forall\mathbf{x}\in\Omega^\circ \tag{2.4}$$

$$(\chi(\boldsymbol\sigma)=1+\varepsilon_{ij}(\mathbf{v})\sigma_{ij})$$

3. THE DUAL PROBLEM WITH ENERGY CONSTRAINTS

We will write twice the specific energy of elastic strain $A(u, u)$ in terms of the stress tensor. Then

$$f(\boldsymbol\sigma)=A_{ijkl}\sigma_{ij}\sigma_{kl}$$
$$\varepsilon_{ij}=A_{ijkl}\sigma_{kl},\ \ f_0=\sigma_0^2/[2\mu(1+\nu)] \tag{3.1}$$

The function χ is linear in σ_{ij}, and the infimum is therefore achieved on the boundary of the feasible stress region, i.e., $f(\boldsymbol\sigma) = f_0$. Denote by ζ the Lagrange multiplier and form the Lagrange function

$$\Phi(\boldsymbol\sigma)=1+\varepsilon_{ij}(\mathbf{v})\sigma_{ij}+\zeta A_{ijkl}\sigma_{ij}\sigma_{kl}$$

From the necessary and sufficient condition for a minimum of $\Phi(\boldsymbol\sigma)$,

$$\partial\Phi/\partial\sigma_{ij}=\varepsilon_{ij}(\mathbf{v})+2\zeta A_{ijkl}\sigma_{kl}=0$$

it follows that

$$\varepsilon_{ij}=-\varepsilon_{ij}(\mathbf{v})/(2\zeta) \tag{3.2}$$

Substituting ε_{ij} from (3.2) into (3.1) and equating the resulting expression to f_0, we obtain

$$\zeta=g(\mathbf{v})/(2f_0),\ \ \inf\chi(\boldsymbol\sigma)=1-g(\mathbf{v})$$
$$g(\mathbf{v})=[f_0f(\boldsymbol\sigma(\mathbf{v}))]^{1/2}$$

From this formula we obtain

$$M_0(\mathbf{v}) = \int_{\Omega_v}[1-g(\mathbf{v})]\,dx-\int_{\Gamma_F}F_iv_i\,d\Gamma \tag{3.3}$$

$$\forall\mathbf{v}\in V(\Omega^\circ),\ \ \Omega_v=\{\mathbf{x}\in\Omega^\circ\,|\,g(\mathbf{v})>1\}$$

Theorem 1. Assume that the function $\mathbf{v}^*\in V(\Omega^\circ)$ exists such that $g(\mathbf{v}^*) = 1$ for every $\mathbf{x}\in\Omega^\circ$, and there is a region $\Omega^*\in O(s,\lambda)$ such that the restriction of $\mathbf{u}^* = -f_0\mathbf{v}^*$ to Ω^* is the solution of the integral identity (1.2) for Ω^*. Then Ω^*, $\mathbf{u}^*$ is an optimal solution of problem (1.1)

Proof. Since the function $\mathbf{v}^*$ satisfies the conditions of the theorem, we have $\Omega_v = \varnothing$ and

$$M_0(\mathbf{v}^*) = -\int_{\Gamma_F}F_iv_i^*\,d\Gamma \tag{3.4}$$

Substituting $\mathbf{u} = \mathbf{u}^*$, $\mathbf{v} = \mathbf{v}^*$ into the integral identity (1.2), we obtain

$$f_0^2\int_{\Omega^*}A(\mathbf{v}^*,\mathbf{v}^*)\,dx=-f_0\int_{\Gamma_F}F_iv_i^*\,d\Gamma \tag{3.5}$$

But $g(\mathbf{v}^*) = [f_0A(\mathbf{v}^*, \mathbf{v}^*)]^{1/2} = 1$, and therefore from (3.4) we obtain

$$\operatorname{mes}\Omega^* = -\int_{\Gamma_F} F_i v_i^* \, d\Gamma$$

whence by (3.3) Ω^*, $\mathbf{u}^*$ is an optimal solution of problem (1.1).

4. DUAL PROBLEMS WITH CONSTRAINTS ON OCTAHEDRAL STRESS

In this case,

$$f(\sigma)=[\sigma_{ij}\sigma_{ij}-\xi I_1^2(\sigma)/3]/(2\mu)$$
$$\xi = \begin{cases} 1, & N=3 \text{ and PSS} \\ 1+2\nu-2\nu^2, & \text{PDS} \end{cases} \tag{4.1}$$

where PSS stands for the plane stress state and PDS for the plane strain state, $I_1(\sigma)$ is the first invariant of the stress tensor, and $f_0 = \sigma_0^2/(3\mu)$. We define the Lagrange function

$$\chi(\sigma)=1+\varepsilon_{ij}(\mathbf{v})\sigma_{ij}+\zeta[\sigma_{ij}\sigma_{ij}-\xi I_1^2(\sigma)/3]$$

From the necessary and sufficient conditions for a minimum of $\chi(\sigma)$ we obtain

$$\varepsilon_{ij}(\mathbf{v})+2\zeta[\sigma_{ij}-\xi I_1(\sigma)\delta_{ij}/3]=0 \tag{4.2}$$

where δ_{ij} is the Kronecker delta. Analysis shows that for $N = 3$ system (4.2) is degenerate. Adding Eqs (4.2) for $i = j$, we obtain

$$I_1(\varepsilon(\mathbf{v}))=0 \tag{4.3}$$

If (4.3) is not satisfied, the function $\chi(\sigma)$has no lower limit. If (4.3) is satisfied for $N = 3$, the system of equations (4.2) is consistent and has the solution

$$\sigma_{ii}=-[\varepsilon_{ii}(\mathbf{v})+\alpha]/(2\zeta), \quad \sigma_{ij}=-\varepsilon_{ij}(\mathbf{v})/(2\zeta) \tag{4.4}$$
$$\alpha = \begin{cases} \text{arbitrary constant for } N=3 \\ I_1(\varepsilon(\mathbf{v})), \text{ for PSS} \\ (1+2\nu-2\nu^2)I_1(\varepsilon(\mathbf{v}))/(1-2\nu)^2, \text{ for PDS} \end{cases}$$

Substituting σ_{ij} from (4.4) into (4.1) and equating the right-hand side to f_0, we find

$$\zeta=g(\mathbf{v})/(2f_0), \quad \inf\chi(\sigma)=1-g(\mathbf{v}) \tag{4.5}$$
$$g(\mathbf{v}) = \begin{cases} [f_0 f(\sigma(\mathbf{v}))]^{1/2}, & N=3 \\ [f_0(\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})+I_1^2(\varepsilon(\mathbf{v}))2\mu]^{1/2}, & \text{PSS} \\ [f_0(\sigma_{ij}(\mathbf{v})\sigma_{ij}(\mathbf{v})+I_1^2(\sigma(\mathbf{v}))(2\mu)^{-1}]^{-1}, & \text{PDS} \end{cases}$$

In this case, $M_0(\mathbf{v})$ is defined by relationships (3.4) and for $N = 3$ the function $\mathbf{v}$ should also satisfy the condition (4.3).

5. DUAL PROBLEMS WITH CONSTRAINTS ON THE MAXIMUM SHEAR STRESS

In this case [5],

$$f(\sigma) = \begin{cases} \max|\sigma_i-\sigma_j|, & N=3 \\ \max\{|\sigma_1-\sigma_2|, \ |\sigma_i|\}, & \text{PSS} \\ 2\mu\max\{|\varepsilon_1-\varepsilon_2|, |\varepsilon_i|\}, & \text{PDS} \end{cases} \tag{5.1}$$

where $\sigma_i(\varepsilon_i)$ are the principal stresses (strains) of the tensor $\sigma(\varepsilon)$, and $f_0 = \sigma_0$.

Represent the function χ in the form

$$\chi(\sigma)=1+\sigma_k\varepsilon_{kk}(\mathbf{v}) \tag{5.2}$$

For $N=3$, the set of principal stresses defined by the inequality $f(\sigma)\leqslant f_0$ is an infinite hexagonal prism making equal angles with the axes σ_k [5]. This set is unbounded, and problem (2.4) therefore has a lower-bounded solution if $\mathbf{v}$ satisfies condition (4.3). The function $\chi(\sigma)$ is linear in σ_k, and the minimum is therefore achieved on one of the six edges

$$\sigma_1=\sigma_3\pm f_0,\ \sigma_2=\sigma_3\pm f_0,\ \chi=1\mp f_0\varepsilon_{33}(\mathbf{v})$$
$$\sigma_1=\sigma_3,\ \sigma_2=\sigma_3\pm f_0,\ \chi=1\pm f_0\varepsilon_{22}(\mathbf{v})$$
$$\sigma_1=\sigma_3\pm f_0,\ \sigma_2=\sigma_3,\ \chi=1\mp f_0\varepsilon_{11}(\mathbf{v})$$

when it follows that

$$\inf\chi(\sigma)=1-f_0\max|\varepsilon_k(\mathbf{v})| \tag{5.3}$$

because the extremal value of $\varepsilon_{kk}(\mathbf{v})$ is equal to one of the principal strains.

For PSS, the set of principal stresses defined by the inequality $f(\sigma)\leqslant f_0$ is a hexagon. The function $\chi(\sigma)$ is linear, and its minimum is therefore achieved at one of the six extreme points

$$\sigma_1=\pm f_0,\ \sigma_2=0,\ \chi=1\mp f_0\varepsilon_{11}(\mathbf{v})$$
$$\sigma_1=\pm f_0,\ \sigma_2=\pm f_0,\ \chi=1\pm f_0 I_1(\varepsilon(\mathbf{v}))$$
$$\sigma_1=0,\ \sigma_2=\pm f_0,\ \chi=1\pm f_0\varepsilon_{22}(\mathbf{v})$$

whence it follows that

$$\inf\chi(\sigma)=1-f_0\max\{|\varepsilon_k(\mathbf{v})|,\ |I_1(\varepsilon(\mathbf{v}))|\} \tag{5.4}$$

For PDS, the set of principal strains defined by the inequality $f(\sigma)\leqslant f_0$ is also a hexagon in the plane of the principal strains. The function χ represented in the form

$$\chi=1+\varepsilon_k\sigma_{kk}(\mathbf{v}) \tag{5.5}$$

is linear in ε_k and its minimum is therefore achieved at one of the six extreme points

$$\varepsilon_1=\pm f_0(2\mu)^{-1},\ \varepsilon_2=0,\ \chi=1\pm f_0\sigma_{11}(\mathbf{v})(2\mu)^{-1}$$
$$\varepsilon_1=\varepsilon_2=\pm f_0(2\mu)^{-1},\ \chi=1\pm f_0 I_1(\sigma(\mathbf{v}))(2\mu)^{-1}$$
$$\varepsilon_1=0,\ \varepsilon_2=\pm f_0(2\mu)^{-1},\ \chi=1\pm f_0\sigma_{22}(\mathbf{v})(2\mu)^{-1}$$

whence we obtain

$$\inf\chi(\sigma)=1-f_0\max\{|\sigma_k(\mathbf{v})|,\ |I_1(\sigma(\mathbf{v}))|\}(2\mu)^{-1} \tag{5.6}$$

Using formulas (5.3), (5.4), (5.6), we define the function

$$g(\mathbf{v})=\begin{cases} f_0\max|\varepsilon_k(\mathbf{v})|, & N=3\\ f_0\max\{|\varepsilon_k(\mathbf{v})|,\ |I_1(\varepsilon(\mathbf{v}))|\}, & \text{PSS}\\ f_0\max\{|\sigma_k(\mathbf{v})|,\ |I_1(\sigma(\mathbf{v}))|\}, & \text{PDS}\end{cases} \tag{5.7}$$

Then $M_0(\mathbf{v})$ is defined by relationships (3.4), and for $N=3$ the function $\mathbf{v}$ should additionally satisfy condition (4.3).

6. DUAL PROBLEMS WITH CONSTRAINTS ON MAXIMUM STRESSES

In this case,

$$f(\sigma)=\max|\sigma_i| \tag{6.1}$$

for $N = 3$, PSS, and PDS for $0 \leq \nu < 1/2$, and $f_0 = \sigma_0$. Solving problem (2.4) for χ represented in the form (5.2), we find that the minimum is achieved at one of 2^N extreme points of the set $f(\boldsymbol{\sigma}) \leq f_0$ and equals

$$\chi(\boldsymbol{\sigma}) = 1 + f_0 \sum_{k=1}^{N} [\pm \varepsilon_{kk}(\mathbf{v})] \tag{6.2}$$

Expression (6.2) is not invariant to the choice of coordinates. We will use the formula [3]

$$\varepsilon_{kk} = \gamma_{ki}^2 \varepsilon_i$$

where γ_{ki} is the cosine of the angle between $\mathbf{e}_k$ and the ith principal direction of ε. Using the estimate

$$\sum_{k=1}^{N} |\varepsilon_{kk}(\mathbf{v})| = \sum_{k=1}^{N} |\gamma_{ki}^2 \varepsilon_i(\mathbf{v})| \leq \sum_{k=1}^{N} \gamma_{ki}^2 |\varepsilon_i(\mathbf{v})| = \sum_{i=1}^{N} |\varepsilon_i(\mathbf{v})|$$

we obtain

$$\inf \chi(\boldsymbol{\sigma}) = 1 - g(\mathbf{v}), \quad g(\mathbf{v}) = f_0 \sum_{i=1}^{N} |\varepsilon_i(\mathbf{v})| \tag{6.3}$$

The dual functional $M_0(\mathbf{v})$ is given by relationships (3.4).

7. DUAL PROBLEMS WITH CONSTRAINTS ON MAXIMUM STRAINS

In this case,

$$f(\mathbf{a} \cdot\cdot \boldsymbol{\varepsilon}) = \max |\varepsilon_i| \tag{7.1}$$

for $N = 3$, PSS, and PDS for $0 \leq \nu < 1/2$, and $f_0 = \sigma_0/[2(1+\nu)\mu]$. Using relationships (7.1) and (5.5) and the arguments and conclusions of Sec. 6, with σ_i and $\varepsilon_i(\mathbf{v})$ replaced, respectively, by ε_i and $\sigma_i(\mathbf{v})$, we obtain

$$g(\mathbf{v}) = f_0 \sum_{i=1}^{N} |\sigma_i(\mathbf{v})| \tag{7.2}$$

The dual functional $M_0(\mathbf{v})$ is defined by relationships (3.4).

Theorem 2. Assume that the function $\mathbf{v}^* \in V(\Omega^\circ)$ exists such that $g(\mathbf{v}^*) \leq 1$, $A(\mathbf{v}^*, \mathbf{v}^*) = \beta$ for every $\mathbf{x} \in \Omega^\circ$. The function $g(\mathbf{v})$ is defined by (4.5), (5.7), (6.3) or (7.2) depending on the strength constraints. Suppose that the region $\Omega^* \in O(s, \lambda)$ exists such that the restriction $\mathbf{u}^* = -\mathbf{v}^*/\beta$ is the solution of the integral identity (1.2) for Ω^*. Then Ω^*, $\mathbf{u}^*$ is an optimal solution of problem (1.1).

Proof. At each point $\mathbf{x} \in \Omega^*$, $\mathbf{v}^*$ satisfies the equality $g(\mathbf{v}^*) = 1$. Thus, $\Omega_\nu = \varnothing$ and

$$M_0(\mathbf{v}^*) = -\int_{\Gamma_F} F_i v_i^* \, d\Gamma \tag{7.3}$$

Substituting $\mathbf{u} = \mathbf{v} = \mathbf{u}^*$ into the integral identity (1.2) for $\Omega = \Omega^*$, we obtain

$$\beta^{-2} \int_{\Omega^*} A(\mathbf{v}^*, \mathbf{v}^*) \, dx = -\beta^{-1} \int_{\Gamma_F} F_i v_i^* d\Gamma \tag{7.4}$$

At each point $\mathbf{x} \in \Omega^0$, $\mathbf{v}^*$ satisfies the equality $A(\mathbf{v}^*, \mathbf{v}^*) = \beta$. Thus, from (7.4) we have

$$\operatorname{mes} \Omega^* = -\int_{\Gamma_F} F_i v_i^* d\Gamma \tag{7.5}$$

Comparing (7.3) and (7.5), we obtain that Ω^*, $\mathbf{u}^*$ is an optimal solution of problem (1.1).

Remark 1. The region $\Omega^* \in O(s, \lambda)$ does not necessarily exist. Yet even if it does not exist, there may exist some infinite-dimensional region where the conditions of the theorem are satisfied. In this case, we can only speak of an achievable bound of the functional (1.1).

Remark 2. In the conditions of the theorem it is implied that the function $\mathbf{v}(\mathbf{x})$ in (4.5) and (5.7) for $N = 3$ satisfies the equality (4.3).

Example 1. Let Ω° be a square of side $2d$, with a uniformly distributed compressive load of strength F applied on two of its edges. We assume that a plane state of strain is realized. Take $v_1^* = \alpha x_1$, $v_2^* = -\alpha x_2$, where α is defined in terms of σ_0 and ν, depending on the stress constraints, from the equation $g(\mathbf{v}^*) = 1$. Substituting $\mathbf{v}^*$ into the functional (3.4), we obtain

$$M_0(\mathbf{v}^*) = 2\int_{-d}^{d} F\alpha d\, d\Gamma = 4\alpha F d^2$$

Example 2. Let Ω° be the region enclosed between two cylinders of radii a and b and length l. A uniformly distributed load F is applied to the outer surface of the cylinder of radius b, normal to the surface. On the surface of the cylinder of radius a, the displacements normal to the surface are zero. Let

$$v_R^* = \alpha\left(\frac{1}{R} - \frac{R}{a^2}\right) v_\varphi^* = 0, \quad v_z^* = \frac{2\alpha}{a^2} z$$

where R, φ and z are cylindrical coordinates. For this vector $\mathbf{v}^*$ we have

$$\varepsilon_R = -\alpha\left(\frac{1}{R^2} + \frac{1}{a^2}\right), \quad \varepsilon_\varphi = \alpha\left(\frac{1}{R^2} - \frac{1}{a^2}\right), \quad \varepsilon_z = \frac{2\alpha}{a^2}$$

and thus $I_1(\varepsilon(\mathbf{v}^*)) = 0$. The constant α is obtained from the inequality $g(\mathbf{v}^*) \leq 1$ depending on ν and σ_0. Substituting $\mathbf{v}^*$ into the functional (3.4), we obtain

$$M_0(\mathbf{v}^*) = -\int_0^{2\pi}\int_0^{l} \alpha F\left(\frac{1}{b} - \frac{b}{a^2}\right) b\, dz\, d\varphi = \frac{2\pi\alpha F(b^2 - a^2) l}{a^2}$$

Example 3. Let Ω° be the parallelepiped $-c < x_1 < c$, $-c < x_2 < c$, $0 < x_3 < d_3$, the surface Γ_u lies in the plane $x_3 = 0$ and is defined by the inequalities $-c < x_1 < c$, $-c < x_2 < c$, and Γ_F° lies in the plane $x_3 = d_3$ and is defined by the inequalities $-d_1/2 < x_1 < d_1/2$, $-d_2/2 < x_2 < d_2/2$. The load on the surface Γ_F is given by $F = F\cos\gamma e_2 + F\sin\gamma e_3$, where F is a positive constant. We assume that $c > d_1 + d_2 + d_3$.

To find the dual estimate, consider the augmented problem, which differs from the previous one by a wider class of functions: $u_1 \neq 0$ on Γ_u°. For this problem, and therefore for the original problem, the dual estimate may be obtained for $v_i^* = \beta_i x_i$, where

$$\beta_1 = -\beta_3, \quad \beta_2 = \alpha\cos\gamma/\Delta, \quad \beta_3 = \alpha\sin\gamma/(4\Delta)$$
$$\Delta = (\cos^2\gamma + 1/4\sin^2\gamma)^{1/2}$$

and α is defined in terms of ν and σ_0, depending on the stress constraints, from the equation $g(\mathbf{v}^*) = 1$. Substituting $\mathbf{v}^*$ into the functional (3.4), we obtain

$$M_0(\mathbf{v}^*) = F d_1 d_2 d_3 \alpha\Delta$$

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